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M5 brane from mass deformed BLG theory

Andreas Gustavsson

Center for quantum spacetime (CQUeST), Sogang University, Seoul 121-742, Korea
School of Physics & Astronomy, Seoul National University, Seoul 151-747 Korea

E-mail: a.r.gustavsson@swipnet.se

Abstract: We study small fluctuations around a BPS three-sphere vacuum solution of mass deformed BLG theory. We realize the BLG theory by a Nambu bracket and find a maximally supersymmetric lagrangian for the fluctuation fields corresponding to a single M5 brane on $\mathbb{R}^{1,2} \times S^3$.

Keywords: Field Theories in Lower Dimensions, Extended Supersymmetry, Chern-Simons Theories, M-Theory

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Contents

1 Introduction

The low energy effective theory living on finitely many coincident M2 branes probing the orbifold singularity $\mathbb{R}^8/\mathbb{Z}_k$ was found in [12]. It is a Chern-Simons theory with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ coupled to matter fields with manifest $\mathcal{N} = 6$ supersymmetry and $\mathrm{SU}(4) \times \mathrm{U}(1)$ R symmetry. For a given gauge group, the only free parameter is the integer valued Chern-Simons level k. For levels $k = 1, 2$ the theory has enhanced $\mathrm{OSp}(8|4)$ maximal superconformal symmetry [12, 17–20].

Subsequently a larger class of $\mathcal{N} = 6$ superconformal theories were found for various gauge groups [13]. We will refer to any $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory as ABJM theory. It is unclear to me whether all these theories correspond to M2 branes probing an $\mathbb{R}^8/\mathbb{Z}_k$ singularity. In any case, for $k = 1, 2$ all these ABJM theories get enhanced $\mathrm{OSp}(8|4)$ superconformal symmetry.

ABJM theories can also be formulated using a particular class of three-algebras [15] called hermitian three-algebras. Another type of three-algebra has been found for the $\mathcal{N} = 5$ supersymmetric theories [16].

The smallest non-trivial ABJM gauge group is $\mathrm{SO}(4)$. For this choice of gauge group, the ABJM lagrangian can be recast in a form that is manifestly $\mathrm{OSp}(8|4)$ invariant, which is then the BLG lagrangian [14] up to a triality of $\mathrm{SO}(8)$ R-symmetry indices.

There are mass deformations of BLG and ABJM theories [13, 21, 26] (older works on mass deformed M2 brane theory are from gravity point of view [27] and from matrix theory point of view [28]) that preserve all the manifest supersymmetries. For ABJM theories this means $\mathcal{N} = 6$ supersymmetry. However the $\mathrm{SO}(6)$ R symmetry is broken by the mass deformation to $\mathrm{SO}(4) \times \mathrm{SO}(2)$. For BLG theory the mass deformation preserves $\mathcal{N} = 8$ supersymmetry and breaks the $\mathrm{SO}(8)$ R symmetry down to $\mathrm{SO}(4) \times \mathrm{SO}(4)$. It is plausible

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that also the above mentioned mass deformed ABJM theories will get enhanced $\mathcal{N}=8$ supersymmetry for levels $k=1,2$, along with an enhanced $\mathrm{SO}(4)\times\mathrm{SO}(4)$ R symmetry.

For levels $k=1,2$ then, we can in mass deformed ABJM and BLG theories, find a vacuum solution which preserves $\mathcal{N}=8$ supersymmetry. Thus

$$\mathbb{R}^{1,2}\times\frac{S^3_{\text{fuzzy}}}{\mathbb{Z}_k}. \tag{1.1}$$

The fuzzy three-sphere is described by four matrices G^i of a certain size $N\times N$ [5]. This construction generalizes the fuzzy two-sphere construction in [1]. In the large N limit we can map these matrices to the embedding functions T^i of a classical three-sphere. These obey the three-algebra and three-sphere constraint

$$\begin{aligned}\{T^i,T^j,T^k\} &= \frac{1}{R}\epsilon^{ijkl}T^l,\\ T^iT^i &= R^2\end{aligned} \tag{1.2}$$

respectively, for a three-sphere of radius R. The curly three-bracket is the Nambu bracket as defined in eq. (2.9). (Our general definition is in eq. (3.14)). Since

$$K^{ij}=\{T^i,T^j,\cdot\} \tag{1.3}$$

are nothing but the six Killing vectors on the three-sphere generating the rotation group $\mathrm{SO}(4)$, we have a realization of the $\mathrm{SO}(4)$ three-algebra which is the smallest non-trivial three-algebra, and in fact the only possible three-algebra of finite dimension (if we assume a few requirements which are all very natural from physics point of view). However there is an infinite dimensional extension of the $\mathrm{SO}(4)$ three-algebra, which is generated by any function on S^3 which has a Taylor series expansion

$$f(T^i)=\sum_{k=1}^{\infty}f_{i_1\ldots i_k}T^{i_1}\ldots T^{i_k}. \tag{1.4}$$

Due to the three-sphere constraint on T^iT^i we only need to consider traceless symmetric tensors $f_{i_1\ldots i_k}$. We could now consider new three-algebra generators

$$T^{i_1\ldots i_k}=T^{i_1}\cdots T^{i_k} \tag{1.5}$$

and we find that all these generate an infinite dimensional three-algebra.

In line with these considerations it is natural to also expect that ABJM theory with gauge group $U(N)\times U(N)$, in the large N limit can be mapped into BLG theory which is realized by a Nambu three-bracket on $S^3/\mathbb{Z}_k$ which should be viewed as an $S^1/\mathbb{Z}_k$ bundle over S^2, so that in particular the large k limit is S^2 [3]. As an aside, since BLG theory is maximally supersymmetric for any k, this means that we should find supersymmetry enhancement in ABJM theory in the large N limit for any level k.

In this paper we will only study BLG theory with a Nambu bracket on S^3. As argued in the paragraph above, this seems to correspond to taking $k=1$ and $N=\infty$ in ABJM theory.

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Fluctuation analysis

In the spirit of [2, 3], we will obtain the induced theory of small fluctuations about the maximally supersymmetric three-sphere vacuum solution of BLG theory. If we temporarily let X collectively denote all the fields in BLG theory, then we expand the mass deformed BLG lagrangian in small fluctuations around the vacuum. We thus write $X = T + \delta X$ where T is the vacuum configuration, and expand the lagrangian as

$$\mathcal{L}(X) = \mathcal{L}(T) + \delta X \frac{\delta \mathcal{L}}{\delta X} + \frac{1}{2}(\delta X)^2 \frac{\delta \mathcal{L}}{\delta X \delta X} + \cdots \tag{1.6}$$

All derivatives are evaluated at T. If T is a static supersymmetric vacuum, then the lagrangian is minus the hamiltonian and this is minimized at the supersymmetric vacuum. Hence the first order derivatives all vanish and we are left with

$$\mathcal{L}(X) = \mathcal{L}(T) + \frac{1}{2}(\delta X)^2 \frac{\delta \mathcal{L}}{\delta X \delta X} + \cdots . \tag{1.7}$$

In a static supersymmetric vacuum we have $\mathcal{L}(T) = 0$ and we need not write out the zeroth order term $\mathcal{L}(T)$. However, $\mathcal{L}(T)(= 0)$ is invariant only under the unbroken supersymmetries. If we do not write out the term $\mathcal{L}(T)$ (since it is zero anyway), then it looks like the supersymmetry variation of the full action can be found be just computing the supersymmetry variation of the second order term. This is wrong. Zero need not be invariant under a variation. We may consider a vacuum in which $\psi = 0$. This does not mean that the supersymmetry variation of ψ must also be $= 0$. In fact the condition for $\delta\psi = 0$ defines in this case the unbroken supersymmetries. Since the higher order terms must cancel the supersymmetry variation of the zeroth order term (because the sum is equal to $\mathcal{L}(X)$ which is the maximally supersymmetric lagrangian), we see that the higher order terms can not be invariant under the broken supersymmetries either. On the other hand, the higher order terms must be invariant under the unbroken supersymmetries since the total lagrangian is invariant, as well as the zeroth order term [10].

Previous work on relating BLG theory with a Nambu three-bracket to M5 brane can be found in [6–9, 11]. In [11] the Carrollian limit of BLG theory (where the speed of light goes to zero) with a Nambu bracket was derived from a single M5 in an infinite tension limit.

Since many calculations in our paper are the same as those in [6], we should contrast those calculations with ours. In [6] the BLG theory is expanded about some background T in which three scalar fields acquire a non vanishing vev. This background does not provide any scale parameter which can be used to perform a systematic fluctuation analysis. Instead the coupling constant $1/k$ in BLG theory must be used as expansion parameter. This means that the strong couling regime of BLG theory can not be treated. The connection between the background T and the internal three-manifold on which the Nambu three-bracket is to be defined, is left unspecified. Naively the background T in these papers appears to be non-supersymmetric. However BLG theory also has a shift symmetry of the fermion. By breaking this shift symmetry one can render the background invariant under modified BLG supersymmetry variations where one has added a constant shift to the variation of the fermion [10]. One may then restore the shift symmetry of the fermion (albeit the fermion now is located at a shifted value) in BLG theory and find that this shift symmetry transmutes into a gauge symmetry (a constant shift proportional to the volume form on the three-manifold) that acts on the background three-form gauge potential C, from the M5 brane brane point of view. For this approach to work one must also specify some condition on the supersymmetry parameter living on the three-manifold. Perhaps this approach can be consistent on a flat three-torus appropriately embedded in transverse space, on which we may have a constant spinor. The connection between the shift symmetry of the BLG fermion and the gauge variation of the constant background C-field could be interesting and worth further study. We note that the M2 brane also couples electrically to C but this field does not seem to alter the BLG theory as long as C is constant, however its field strength dC has the effect of mass deforming BLG theory [26].

In this paper we instead follow the approach of [2, 3]. We expand about a maximally supersymmetric three-sphere vacuum solution in mass deformed BLG theory. This background provides us with a mass parameter that we can use to quantify the smallness of our fluctuation fields. Hence we can have a small value on k and still have a sensible fluctuation expansion by having a small mass parameter as expansion parameter. Since the background does not break any supersymmetry we find a maximally supersymmetric M5 brane theory on a three-sphere.

The theory of a single M5 brane is subtle due to the selfdual three-form. On a topologically non-trivial space-time one can find several different quantum theories of the selfdual three-form [4]. Consequently the lagrangian of the selfdual three-form can not be unique, but there must be one lagrangian for each such theory. It seems plausible that this is related to the fact that one can not write down a manifestly covariant lagrangian [22]. Then it could be that by making a large diffeormorphism (a diffeomorphism not continuously connected to the identity) one transforms one lagrangian into another.

2 Infinite-dimensional mass-deformed BLG theory

Our starting point will be BLG theory, realized by a Nambu bracket on some internal three-manifold M_3. One may attempt to define the Nambu bracket using an auxilary three-manifold on which we have a constant supersymmetry parameter transforming as a scalar. But if M_3 is not embedded in eleven-dimensional space-time and given the uniqueness of M theory as the only consistent quantum theory in eleven dimensions, it seems plausible that such a BLG theory would become inconsistent at the quantum level. The next simplest example is to take M_3 to be a flat three-torus embedded in transverse eight-dimensional space on which we again can have a constant supersymmetry parameter which now transforms as a spinor due to the R-symmetry index being a Weyl spinor of SO(8). The scalar fields X^I are now functions of three parameters θ^α that parametrize M_3 embedded in $\mathbb{R}^8$ as

$$\theta^\alpha \mapsto X^I(\theta). \tag{2.1}$$

Then we can at least locally always choose the three coordinates θ^α to coincide with some three of the eight scalar fields,

$$X^\alpha = \theta^\alpha. \tag{2.2}$$

From purely geometric considerations we have now obtained a non-vanishing vacuum expectation value of three of the scalar fields! This background is not obviously supersymmetric, but can presumably be made supersymmetric by also turning on a holonomy on M_3. This example with flat torus embedded in transverse space is the first example one naturally thinks of if one asks how the Nambu bracket should be defined. This example was analyzed in [6] though there appears to be a few loose ends that are yet to be understood. It is not very clear how to assure that fluctuations are governed by a supersymmetric theory, and seems to require that one turns on a holonomy on M_3. It is not clear if this theory arises in the large rank limit of ABJM theory when expanded about some vacuum in ABJM theory.

A less obvious choice of Nambu bracket and M_3 is a three-sphere. However this is the most natural choice if one instead asks what is a supersymmetric vacuum about which one can expand BLG theory such that one gets a supersymmetric theory for the fluctuations. Also it is fairly natural to expect that the three-sphere arises as the large rank limit of the fuzzy three-sphere vacuum solution in mass deformed ABJM theory at level $k = 1$. Since the three-sphere preserves maximal supersymmetry there is no issue regarding how to maintain supersymmetry in the fluctuation analysis about the three-sphere.

The fuzzy three-sphere is more complicated than the fuzzy two-sphere. It can therefore be useful to make a comparison between the fuzzy two-sphere and the fuzzy three-sphere. The fuzzy two-sphere is defined in terms of generators of SU(2) in some $N+1$ dimensional representation say, where N can be any positive integer. In the large N limit we can map these SU(2) generators into the three Killing vectors K^i on S^2. These Killing vectors in turn, can be expressed in terms of the Poisson bracket as

$$K^i = \{T^i, \cdot\} \tag{2.3}$$

where T^i describes the embedding of the two-sphere into $\mathbb{R}^3$. The Poisson bracket is defined using the metric on the two-sphere. These K^i obey the SU(2) algebra as a consequence of the Jacobi identity.

The obvious generalization to the three-sphere is that in the large N limit, the fuzzy three-sphere generators are mapped into coordinates T^i that describe the embedding of the three-sphere into $\mathbb{R}^4$. The six Killing vectors on the three-sphere are

$$K^{ij} = \{T^i, T^j, \cdot\} \tag{2.4}$$

The Nambu bracket is defined using the metric on the three-sphere. The Killing vectors then generate the SO(4) Lie algebra as a consequence of the fundamental identity and eq. (1.2). We note that even though the definition of the discrete three-bracket and matrix three-algebra generators in ABJM theory has been obtained explicitly [15], it is more subtle to understand the SO(8) R symmetry in terms of this three-bracket at level $k = 1$. This necessarily requires proper understanding of monopole operators. Using these monopole operators we have found that the ABJM three-bracket becomes essentially totally antisymmetric [17]. This is a promising property if it is to be mapped into a totally antisymmetric Nambu bracket in the large N limit. But due to the complication of having to involve monopole operators, we have not yet obtained a rigorous way of taking the large

N limit of the ABJM theory three-bracket. Taking this large N limit in a rigorous way will be very interesting and we believe that this will eventually lead to an understanding of the theory of multiple M5 branes.

In order to allow for other supersymmetric vacua apart from the three-sphere of mass deformed theory, we will in the rest of this section assume a more generic vacuum three-manifold and denote its embedding in transverse space as $\theta^\alpha \mapsto T^I(\theta)$. We denote Minkowski coordinates on $\mathbb{R}^{1,2}$ as x^μ. We introduce normal coordinates x^A $(A = 1, \dots, 5)$ to M_3 in $\mathbb{R}^8$. We consider the change of coordinates in $\mathbb{R}^8$

$$(\theta^\alpha, x^A) \mapsto x^I = x^I(\theta, x^A). \tag{2.5}$$

The submanifold M_3 is located at constant values of x^A, that we can set to $x^A = 0$ so that

$$T^I(\theta) = x^I(\theta, x^A = 0) \tag{2.6}$$

defines a parametrization of M_3. The induced metric on M_3 is given by

$$g_{\alpha\beta} = \partial_\alpha T^I \partial_\beta T^I. \tag{2.7}$$

We will also need

$$\begin{aligned} g_{AB} &= \partial_A T^I \partial_B T^I, \\ g_{A\alpha} &= 0 \end{aligned} \tag{2.8}$$

on M_3. We define the Nambu bracket of three scalar functions f, g and h on M_3 as

$$\begin{aligned} \{f, g, h\} &= \sqrt{g}\epsilon^{\alpha\beta\gamma}\partial_\alpha f \partial_\beta g \partial_\gamma h \\ &= *(df \wedge dg \wedge dh). \end{aligned} \tag{2.9}$$

We use the convention that

$$\epsilon_{123} = 1 \tag{2.10}$$

and all indices are rised by the inverse metric $g^{\alpha\beta}$. Here the star $*$ denotes the Hodge dual on M_3.

It is very important to stress that the Nambu bracket is calculated with respect to a background metric associated to a vacuum state. Hence the supersymmetry variation of the metric is zero. In that sense, BLG theory with a Nambu bracket, appears to make no sense unless one specifies a non-vanishing vacuum field configuration $X^I = T^I$.[1] It is true that the Nambu bracket always satisfies the fundamental identity on any auxiliary three-manifold. However this is not enough to insure supersymmetry. When checking closure of supersymmetry one needs to make a second supersymmetry variation of the fermion. This will involve a term

$$\{\bar{\epsilon}\Gamma^I\psi, X^J, X^K\}. \tag{2.11}$$

[1] There is the possibility of taking M_3 to be outside eleven dimensional spacetime and this gives a BLG theory at classical level with all the right symmetries, but we do not believe this BLG theory can be consistent at the quantum level given the uniqueness of M theory.

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In order to secure on-shell closure one needs to be able to rewrite this as

$$\bar{\epsilon}\Gamma^I\{\psi, X^J, X^K\}. \tag{2.12}$$

The same type of problem arises when checking supersymmetry of the BLG action. In both cases one has to be able to freely move out the supersymmetry parameter outside the Nambu bracket. On a generic three-manifold it is not possible to have a covariantly constant spinor. This means we can not obtain a supersymmetric BLG theory if we define our Nambu bracket on a generic three-manifold despite the Nambu bracket obeys the fundamental identity.

We introduce a complete set of functions $T^a(\theta)$ on M_3 that will be our generators for the infinite-dimensional three-algebra. We expand the matter fields as

$$\begin{aligned} X^I(x,\theta) &= X^I_a(x)T^a(\theta), \\ \psi(x,\theta) &= \psi_a(x)T^a(\theta) \end{aligned} \tag{2.13}$$

We define the gauge covariant derivative as

$$D_\mu X^I = \partial_\mu X^I - A_{\mu,ab}\{T^a, T^b, X^I\}. \tag{2.14}$$

We use eleven-dimensional spinor notation since we wish to treat $\mathbb{R}^{1,2}$ and M_3 on the same footing, and eventually identify $\mathbb{R}^{1,2} \times M_3$ as the world-volume of M5 brane. The supersymmetry parameter ϵ and spinor field ψ are subject to the chirality conditions

$$\begin{aligned} \tilde{\Gamma}\epsilon &= \epsilon, \\ \tilde{\Gamma}\psi &= -\psi \end{aligned} \tag{2.15}$$

where

$$\tilde{\Gamma} = \Gamma_{012}. \tag{2.16}$$

We have the following $\mathcal{N} = 8$ supersymmetry variations

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\psi \\ \delta\psi &= \Gamma^\mu\Gamma_I\epsilon D_\mu X^I - \frac{1}{6}\Gamma_{IJK}\epsilon\{X^I, X^J, X^K\}, \\ \delta A_{\mu,ab} &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_{[a}\psi_{b]}. \end{aligned} \tag{2.17}$$

closing on-shell,

$$\Gamma^\mu D_\mu\psi + \frac{1}{2}\Gamma_{IJ}\{X^I, X^J, \psi\} = 0, \tag{2.18}$$

$$D^2 X_I - \frac{i}{2}\{\bar{\psi}, \Gamma_{IJ}X^J, \psi\} - \frac{\partial V}{\partial X^I} = 0, \tag{2.19}$$

$$F_{\mu\nu,ab} + \epsilon_{\mu\nu\lambda}\left(X_{I[a}D^\lambda X^I_{b]} + \frac{i}{2}\bar{\psi}_{[a}\Gamma^\lambda\psi_{b]}\right) = 0. \tag{2.20}$$

Here

$$V = \frac{1}{12}\left\langle\{X^I, X^J, X^K\}, \{X^I, X^J, X^K\}\right\rangle \tag{2.21}$$

and the trace form is defined as

$$\langle F,G\rangle = \int d^3\theta\sqrt{g}FG. \tag{2.22}$$

The matter part of the lagrangian density is

$$\begin{aligned}\mathcal{L}_{\text{matter}} &= -\frac{1}{2}\left\langle D_\mu X^I, D^\mu X^I\right\rangle - V \\ &+\frac{i}{2}\left\langle\bar{\psi},\Gamma^\mu D_\mu\psi\right\rangle + \frac{i}{4}\left\langle\bar{\psi},\{\Gamma^{IJ}\psi,X^I,X^J\}\right\rangle.\end{aligned} \tag{2.23}$$

The gauge field part is given by the Chern-Simons term,

$$\mathcal{L}_{\text{CS}} = \frac{1}{2}\epsilon^{\mu\nu\lambda}A_{\mu,ab}\partial_\nu A_{\lambda,cd}\left\langle T^a,\{T^b,T^c,T^d\}\right\rangle + \cdots \tag{2.24}$$

The cubic interaction term (denoted by the ellipses) in the Chern-Simons action will not be of any interest to us in this paper.

Mass deformation

There is a mass deformation of BLG theory [21] which does not break any of the supersymmetries, though it breaks conformal invariance by the introduction of a mass parameter m. It also breaks SO(8) R-symmetry to SO(4) $\times$ SO(4). The embedding of SO(4) $\times$ SO(4) in SO(8) is such that $8_v \to 4_v + 4_v$. Accordingly we split the vector index I as $(i,\hat{i})$. The mass deformed BLG supersymmetry variations are obtained by modifying the variation of the fermion by adding the term

$$\delta'\psi = m\Gamma\Gamma_I\epsilon X^I \tag{2.25}$$

Here

$$\Gamma = \frac{1}{24}\epsilon^{ijkl}\Gamma_{ijkl}. \tag{2.26}$$

To maintain a maximally supersymmetric lagrangian, we add the following terms to the lagrangian [21]

$$\begin{aligned}\mathcal{L} &= -\frac{m^2}{2}\left\langle X^I,X^I\right\rangle - \frac{im}{2}\left\langle\bar{\psi},\Gamma\psi\right\rangle \\ &-\frac{m}{6}\left(\epsilon_{ijkl}\left\langle X^i,\{X^j,X^k,X^l\}\right\rangle + \epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}\left\langle X^{\hat{i}}\{X^{\hat{j}},X^{\hat{k}},X^{\hat{l}}\}\right\rangle\right).\end{aligned} \tag{2.27}$$

In a background with $\psi = 0$, the non-trivial condition for unbroken supersymmetry is that

$$\delta\psi = 0. \tag{2.28}$$

Assuming that only the four scalar fields X^i are excited and $X^{\hat{i}} = 0$, the condition for unbroken supersymmetry, in a static field configuration, reads

$$0 = \left(mX^i + \frac{1}{6}\epsilon_{ijkl}\{X^j,X^k,X^l\}\right)\Gamma_i\epsilon \tag{2.29}$$

This condition does not restrict the supersymmetry parameter. We can write the condition for the maximally supersymmetric vacuum field configuration as

$$\{X^i, X^j, X^k\} = m\epsilon^{ijkl}X^l. \tag{2.30}$$

We can solve this equation by taking

$$X^i X^i = \frac{1}{m^2}. \tag{2.31}$$

that is we find a three-sphere of radius

$$R = \frac{1}{m}. \tag{2.32}$$

3 Constant spinor and the Nambu bracket

In order to have closure of the BLG supersymmetry variations we must require that the supersymmetry parameter ϵ is such that

$$\{\epsilon, X^I, X^J\} = 0 \tag{3.1}$$

This condition comes from taking a second supersymmetry variation on the fermion and demanding on-shell closure. Clearly we must extend our definition of the Nambu bracket to the case where the entries are not scalar entities.

In ABJM theory with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ say, for any finite N, apparently the supersymmetry parameter ϵ is just a constant,

$$\partial_M \epsilon = 0. \tag{3.2}$$

However this equation is not covariant, and is written in flat eleven-dimensional Minkowski coordinates x^M. We can write the condition in a covariant way as

$$D_M\epsilon = (\partial_M + \Omega_M)\epsilon = 0 \tag{3.3}$$

where Ω_M is the spin connection. In the infinite N limit we have a classical three-sphere and it is then more useful to express the constancy condition in terms of the polar coordinates

$$x^M = x^M(x^\mu, \theta^\alpha, R, x^{\hat{i}}) \tag{3.4}$$

for which the metric is given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + g_{\alpha\beta}d\theta^\alpha d\theta^\beta + dR^2 + dx^{\hat{i}}dx^{\hat{i}} \tag{3.5}$$

and we find the following non-vanishing Christoffel symbols,

$$\begin{aligned} \left(A^T_\alpha\right)^\gamma{}_\beta &= \Gamma^\gamma_{\alpha\beta} \\ \left(A^N_\alpha\right)^R{}_\beta &= -\frac{g_{\alpha\beta}}{R} \end{aligned} \tag{3.6}$$

which we interpret as two gauge fields associated with the tangent bundle and the normal bundle of the three-sphere respectively. In terms of these coordinates the Killing spinor equation pulled back to the three-sphere reads[2]

$$D_\alpha \epsilon = \left(D_\alpha^T - \frac{1}{2R}\Gamma_R \Gamma_\alpha\right)\epsilon = 0 \tag{3.10}$$

where $D_\alpha^T \equiv \partial_\alpha + A_\alpha^T$ is the intrinsic covariant derivative on the three-sphere. Note that, since $\mathbb{R}^{1,10}$ is flat, we have

$$[D_\alpha, D_\beta] = 0. \tag{3.11}$$

The Killing spinor equation $D_\alpha \epsilon = 0$ means that we should define the Nambu bracket as

$$\{\epsilon, X^I, X^J\} = \sqrt{g}\epsilon^{\alpha\beta\gamma} D_\alpha \epsilon \partial_\beta X^I \partial_\gamma X^J. \tag{3.12}$$

This definition is crucial for getting closure of the $\mathcal{N} = 8$ supersymmetry. We note that

$$D_\alpha X^I = \frac{\partial X^I}{\partial \theta^\alpha} D_J X^I = \partial_\alpha X^I \tag{3.13}$$

upon taking the pull back to the three-sphere. It is a bit surprising that a covariant derivative can act on the X^I just as if these were scalar fields since these do actually carry an R symmetry index and accordingly should rather be viewed as a section of an SO(8) vector bundle over S^3. Let us therefore re-derive this 'scalar' field property of the X^I also in an intrinsic way, from the point of view of the three-sphere. On the three-sphere the only relevant non-vanishing Christoffel symbol $\Gamma^R_{\alpha\beta}$ couples to X^β as $D_\alpha X^I = \partial_\alpha X^I + \Gamma^R_{\alpha\beta} X^\beta$. Since there is no field component X^β in BLG theory we again conclude that $D_\alpha X^I = \partial_\alpha X^I$.

The general definition of the Nambu bracket must then be

$$\{f, g, h\} = *(Df \wedge Dg \wedge Dh) \tag{3.14}$$

where D denotes the covariant exterior derivative (including the normal bundle gauge field.[3]).

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[2] In our conventions

$$D_\alpha = \partial_\alpha + \frac{1}{2}\Gamma_{\alpha ab} M^{ab} \tag{3.7}$$

where a is a local flat index and the SO(1,10) algebra generators M^{ab} are normalized so that

$$[M_{ab}, M^{cd}] = -4\delta_{[a}^{[c} M^{b]}{}_{d]} \tag{3.8}$$

In vector and spinor representations we then find

$$\begin{aligned}(M^{ab})_{cd} &= 2\delta^{ab}_{cd},\\ M^{ab} &= \frac{1}{2}\Gamma_{ab}.\end{aligned} \tag{3.9}$$

Here $\Gamma_{\alpha ab} = \eta_{ac}\Gamma^c_{\alpha b} = -\Gamma_{b\alpha a}$ is the Ricci rotation coefficient, or the Christoffel symbol with two indices converted into flat indices by means of two vielbeins.

[3] I would like to thank Soo-Jong Rey for pointing out to me that one has to take into account the normal bundle gauge field

The real three-algebra is defined by a real three-bracket $[\cdot,\cdot,\cdot]$ satisfying the fundamental identity. The three-bracket is totally antisymmetric. We also require the existence of a positive definite trace form $\langle\cdot,\cdot\rangle$ subject to the invariance condition

$$\left\langle [T^c, T^d, T^a], T^b \right\rangle + \left\langle T^a, [T^c, T^d, T^b] \right\rangle = 0. \tag{3.15}$$

The only finite-dimensional example is SO(4). We also have infinite-dimensional algebras realized by the Nambu three-bracket.

We define the associated trace form as

$$\langle f, g\rangle = \int d^3\theta \sqrt{g} f g. \tag{3.16}$$

We can expand any function in a complete basis of functions. We denote the basis elements by

$$T^a = T^a(\theta). \tag{3.17}$$

However, tensoring this basis element by an θ^α-independent spinor or tensor, it is essential to use the total covariant derivative D_α acting on the quantity. For instance we expand the BLG spinor in this three-algebra basis as

$$\psi(x,\theta) = \psi_a(x) T^a(\theta) \tag{3.18}$$

and compute its derivative as

$$D_\alpha \psi(x,\theta) \tag{3.19}$$

even though, of course T^a is just a scalar entity, the $\psi_a(x)$ part carries R-symmetry indices associated both with space-time, internal three-manifold, and its normal bundle as embedded in eleven-dimensions. It is therefore essential that we use the total covariant derivative acting on $\psi(x,\theta)$. However, on the basis elements T^a we act with the ordinary derivative $\partial_\alpha T^a$ since the basis functions are scalar quantities that carry no R symmetry indices nor spacetime indices.

We note that the fundamental identity

$$\{T^a, T^{[b}, \{T^c, T^d, T^{e]}\}\} = 0 \tag{3.20}$$

is satisfied only if we can use ordinary commuting derivatives. We expand the left-hand side (defining $g\epsilon^{\alpha\beta\gamma}\epsilon^{\alpha'\beta'\gamma'} = 6g^{\alpha\beta\gamma,\alpha'\beta'\gamma} = g^{\alpha\alpha'}g^{\beta\beta'}g^{\gamma\gamma'} \pm$anti-symmetric.)

$$\begin{aligned}
g\epsilon^{\alpha\beta\gamma}\epsilon^{\alpha'\beta'\gamma'} & D_\alpha T^a D_\beta T^{[b} D_\gamma \left(D_{\alpha'} T^c D_{\beta'} T^d D_{\gamma'} T^{e]} \right) \\
&= 6 D_\gamma \left(g^{\alpha\beta\gamma,\alpha'\beta'\gamma'} D_\alpha T^a D_\beta T^{[b} D_{\alpha'} T^c D_{\beta'} T^d D_{\gamma'} T^{e]} \right) \\
&\quad -6 g^{\alpha\beta\gamma,\alpha'\beta'\gamma'} (D_\gamma D_\alpha T^a) D_\beta T^{[b} D_{\alpha'} T^c D_{\beta'} T^d D_{\gamma'} T^{e]} \\
&\quad -6 g^{\alpha\beta\gamma,\alpha'\beta'\gamma'} D_\alpha T^a (D_\gamma D_\beta T^{[b}) D_{\alpha'} T^c D_{\beta'} T^d D_{\gamma'} T^{e]}.
\end{aligned} \tag{3.21}$$

We see that the last line vanishes only if the derivatives commute.

We next check the trace invariance condition,

$$
\begin{aligned}
&\left\langle \{T^c, T^d, T^a\}, T^b \right\rangle + \left\langle T^a, \{T^c, T^d, T^b\} \right\rangle \\
&\quad = \int d^3\theta g \epsilon^{\alpha\beta\gamma} \left(D_\alpha T^c D_\beta T^d D_\gamma T^a T^b + T^a D_\alpha T^b D_\beta T^c D_\gamma T^d \right) \\
&\quad = \int d^3\theta g \epsilon^{\alpha\beta\gamma} D_\alpha T^c D_\beta T^d D_\gamma \left(T^a T^b \right)
\end{aligned} \tag{3.22}
$$

This vanishes only if we can write this as a total derivative. This will be the case in all cases we will be interested in. This is so because we use the trace form only to get the lagrangian. Since the lagrangian does not carry any indices we find that the total derivative is an ordinary derivative. Though if we act by an ordinary derivative on a contraction of two spinors for example, we find two covariant derivatives as

$$
\partial_\alpha \left(\bar{\psi}\psi \right) = D_\alpha \bar{\psi}\psi + \bar{\psi} D_\alpha \psi. \tag{3.23}
$$

Since

$$
D_\alpha X^I = \partial_\alpha X^I \tag{3.24}
$$

we find that

$$
\{X^I, X^J, X^K\} = \sqrt{g} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^I \partial_\beta X^J \partial_\gamma X^K \tag{3.25}
$$

and we then find that the fundamental identity holds for these scalar fields,

$$
\{\{X^{[I}, X^J, X^K\}, X^{L]}, X^M\} = 0 \tag{3.26}
$$

This is enough to ensure supersymmetry.

4 Computing the induced Lagrangian

On the M5 brane we have a selfdual three-form which implies that there is no diffeomorphism invariant classical lagrangian formulation of the theory. However by giving up diffeomorphism invariance, we can find a lagrangian description. One example is given in [22] associated to the split of six dimensions into five plus one. Here we find a different version of such a diffeomorphism non-invariant lagrangian, associated to the split of six as three plus three. This lagrangian was also studied in [7].

We aim at finding a six dimensional lagrangian by expanding mass deformed BLG theory about the three-sphere vacuum. We want this six dimensional theory to possess as much diffeomorphism symmetry as possible.

Six-dimensional fluctuation fields

The eight scalar fields correspond to fluctuations in eight dimensional transverse space. As we have already mentioned, we find it convenient to change coordinates as

$$
x^I \mapsto x^I(\theta^\alpha, x^A) \tag{4.1}
$$

Then the three-sphere is a level curve, which we may choose to be located at $x^A = 0$,

$$T^I(\theta) = x^I(\theta, 0) \tag{4.2}$$

We then consider small fluctuations of this three-sphere

$$\delta x^I(\theta, 0) = \delta\theta^\alpha \partial_\alpha x^I(\theta, 0) + \delta x^A \partial_A x^I(\theta, 0) \tag{4.3}$$

For notational convenience, we define

$$Y^I(x, \theta) \equiv \delta x^I(x, \theta, 0) \equiv X^I(x, \theta) - T^I(x, \theta) \tag{4.4}$$

where we re-instated the x^μ dependence as well, to illustrate that these are really six-dimensional fields. We associate six-dimensional fields to these fluctuations as

$$\begin{aligned} \delta\theta^\alpha &= \phi^\alpha \\ \delta x^A &= \phi^A \end{aligned} \tag{4.5}$$

As it turns out, the dual field $B_{\alpha\beta}$ defined as

$$\phi^\alpha = \frac{1}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma} B_{\beta\gamma} \tag{4.6}$$

will be identified as components of a gauge potential in the M5 brane.

We define the remaining gauge field components $B_{\mu\alpha}$ as

$$A_{\mu,ab} T^a \partial_\alpha T^b = B_{\mu\alpha} \tag{4.7}$$

It is not clear whether this relation can be inverted so as to express $A_{\mu,ab}$ in terms of $B_{\mu\alpha}$. Since our goal is to derive the M5 from M2 we will not need to invert this relation for our immediate purposes. However if we were to derive M2 from M5 it seems we would need to invert this relation.

We first show that $B_{\alpha\beta}$ and $B_{\mu\alpha}$ defined as above can really be identified as components of a two-form gauge potential in a six-dimensional theory, by showing that a gauge variation in BLG theory induces a gauge variation of these two-form components. A gauge transformation in BLG theory is given by

$$\begin{aligned} \delta X^I &= \Lambda_{ab}(x)\{T^a, T^b, X^I\}, \\ \delta A_{\mu,ab} &= D_\mu \Lambda_{ab}(x). \end{aligned} \tag{4.8}$$

To linear order we find the induced gauge variations

$$\begin{aligned} \delta B_{\alpha\beta} &= \partial_\alpha \Lambda_\beta - \partial_\beta \Lambda_\alpha, \\ \delta B_{\mu\alpha} &= \partial_\mu \Lambda_\alpha - \partial_\alpha \Lambda_\mu, \\ \delta\phi^A &= 0, \end{aligned} \tag{4.9}$$

with gauge parameter

$$\begin{aligned} \Lambda_\alpha &= \Lambda_{ab} T^a \partial_\alpha T^b, \\ \Lambda_\mu &= 0. \end{aligned} \tag{4.10}$$

We note that

$$D_\alpha B_{\mu\beta} = D^T_\alpha B_{\mu\beta} \tag{4.11}$$

since there is no component $B_{\mu R}$. Also, then we have as usual that

$$D_\alpha B_{\mu\beta} - D_\beta B_{\mu\alpha} = \partial_\alpha B_{\mu\beta} - \partial_\beta B_{\mu\alpha}. \tag{4.12}$$

If we define

$$B_{\alpha\mu} = -B_{\mu\alpha} \tag{4.13}$$

then we define the field strength components as

$$\begin{aligned} H_{\mu\alpha\beta} &= \partial_\mu B_{\alpha\beta} + \partial_\alpha B_{\beta\mu} - \partial_\beta B_{\alpha\mu}, \\ H_{\alpha\beta\gamma} &= \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\gamma B_{\alpha\beta}. \end{aligned} \tag{4.14}$$

To show that the action is supersymmetric we need Bianchi identities

$$\begin{aligned} D_{[\alpha} H_{\beta\gamma\delta]} &= 0 \\ D_{[\alpha} H_{\mu\beta\gamma]} &= 0 \\ D_{[\alpha} H_{\mu\nu\beta]} &= 0 \end{aligned} \tag{4.15}$$

where we define

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} - \partial_\nu B_{\mu\alpha}. \tag{4.16}$$

The supersymmetry parameter ϵ and spinor field ψ in BLG theory are subject to chirality conditions

$$\begin{aligned} \tilde{\Gamma}\epsilon &= \epsilon \\ \tilde{\Gamma}\psi &= -\psi. \end{aligned} \tag{4.17}$$

These Weyl conditions are not six-dimensional. The 'chirality matrix' associated with the three-manifold is given by

$$\begin{aligned} \Sigma &= \frac{1}{6}\Gamma_{IJK}\{T^I, T^J, T^K\} \\ &= \frac{1}{6}\sqrt{g}\epsilon^{\alpha\beta\gamma}\Gamma_{\alpha\beta\gamma}. \end{aligned} \tag{4.18}$$

This matrix has the anti-properties

$$\begin{aligned} \Sigma^\dagger &= -\Sigma, \\ \Sigma^2 &= -1 \end{aligned} \tag{4.19}$$

but when combined with the SO(8) chirality matrix $\tilde{\Gamma}$, we find a true (six-dimensional) chirality matrix

$$\tilde{\Gamma}\Sigma. \tag{4.20}$$

We would like to find new spinors ω and χ respectively, such that these are subject to the Weyl conditions

$$
\begin{aligned}
\tilde{\Gamma}\Sigma\omega &= -\omega \\
\tilde{\Gamma}\Sigma\chi &= \chi
\end{aligned}
\tag{4.21}
$$

which are of a six-dimensional covariant form. We find these conditions by making the unitary rotation

$$
\begin{aligned}
\epsilon &= U\omega \\
\psi &= U\chi
\end{aligned}
\tag{4.22}
$$

with

$$
U = \frac{i}{\sqrt{2}}\tilde{\Gamma}(1-\Sigma). \tag{4.23}
$$

Scalar matter part. The scalar matter field part is

$$
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} \tag{4.24}
$$

where

$$
\begin{aligned}
\mathcal{L}_{\text{kin}} &= -\frac{1}{2}\left\langle D^{\mu}X^{I}, D_{\mu}X^{I}\right\rangle \\
\mathcal{L}_{\text{pot}} &= -\frac{1}{12}\left\langle \{X^{I},X^{J},X^{K}\},\{X^{I},X^{J},X^{K}\}\right\rangle - \frac{m^2}{2}\left\langle X^{I},X^{I}\right\rangle \\
&\quad -\frac{m}{6}\epsilon^{ijkl}\left\langle X^{i}\{X^{j},X^{k},X^{l}\}\right\rangle - \frac{m}{6}\epsilon^{\hat{i}\hat{j}\hat{k}\hat{l}}\left\langle X^{\hat{i}}\{X^{\hat{j}},X^{\hat{k}},X^{\hat{l}}\}\right\rangle .
\end{aligned}
\tag{4.25}
$$

To zeroth order in fluctuation fields, we find that

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{12}\left\langle \{T^{i},T^{j},T^{k}\},\{T^{i},T^{j},T^{k}\}\right\rangle - \frac{m^2}{2}\left\langle T^{i},T^{i}\right\rangle - \frac{m}{6}\epsilon^{ijkl}\left\langle T^{i},\{T^{j},T^{k},T^{l}\}\right\rangle \\
&= -\frac{1}{2} - \frac{1}{2} + 1 = 0.
\end{aligned}
\tag{4.26}
$$

This being zero reflects the fact that the three-sphere solution is a supersymmetric ground state.

To first order we find

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\left\langle \{T^{i},T^{j},T^{k}\},\{T^{i},T^{j},Y^{k}\}\right\rangle - m^2\left\langle T^{i},Y^{i}\right\rangle - \frac{2m}{3}\epsilon^{ijkl}\left\langle Y^{i},\{T^{j},T^{k},T^{l}\}\right\rangle \\
&= (-3+4-1)m^2\left\langle T^{i},Y^{i}\right\rangle = 0.
\end{aligned}
\tag{4.27}
$$

This being zero means that the three-sphere is a solution to the classical equation of motion.

The first non-vanishing contributions starts at quadratic order. There will be higher order corrections but these are suppressed by an order of $1/R$ and can be ignored by taking R sufficiently large. In this paper we will compute only up to quadratic order.

We start by computing the kinetic term. First we compute

$$\begin{aligned} D_\mu X^i &= \frac{1}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma}H_{\mu\alpha\beta}\partial_\gamma T^i + \partial_\mu\phi^R\partial_R T^i, \\ D_\mu X^{\hat{i}} &= \partial_\mu\phi^{\hat{i}}. \end{aligned} \tag{4.28}$$

and consequently we get

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}H_{\mu\alpha\beta}H^{\mu\alpha\beta} - \frac{1}{2}\partial_\mu\phi^A\partial^\mu\phi^A. \tag{4.29}$$

We next expand the potential term,

$$\begin{aligned} \mathcal{L}_{\rm pot} = &-\frac{1}{2}\Big\langle\{Y^i,Y^j,T^k\},\{T^i,T^j,T^k\}\Big\rangle \\ &-\frac{1}{2}\Big\langle\{T^i,T^j,Y^k\},\{Y^i,T^j,T^k\}\Big\rangle \\ &-\frac{1}{4}\left\langle\{T^i,T^j,Y^K\},\{T^i,T^j,Y^K\}\right\rangle \\ &-\frac{m^2}{2}\left\langle Y^I,Y^I\right\rangle - m\epsilon^{ijkl}\Big\langle Y^i,\{Y^j,T^k,T^l\}\Big\rangle. \end{aligned} \tag{4.30}$$

We may use the trace invariance condition and the fundamental identity and get the identity

$$\langle\{a,b,c\},\{e,f,g\}\rangle = 3\langle\{f,g,c\},\{e,a,b\}\rangle \tag{4.31}$$

where the right-hand side is to be antisymmetrized in e,f,g. Using this we can bring the lagrangian into the form of a sum of two terms,

$$\mathcal{L}_{\rm pot} = \mathcal{L}_I + \mathcal{L}_m \tag{4.32}$$

where

$$\begin{aligned} \mathcal{L}_I &= \frac{1}{2}\Big\langle\{T^j,T^k,Y^k\},\{T^j,T^i,Y^i\}\Big\rangle - \frac{1}{4}\left\langle\{T^i,T^j,Y^K\},\{T^i,T^j,Y^K\}\right\rangle \\ \mathcal{L}_m &= -\frac{m^2}{2}\left\langle Y^I,Y^I\right\rangle. \end{aligned} \tag{4.33}$$

We then note that[4]

$$g^{\alpha\beta}\partial_\alpha T^i\partial_\beta T^j = \delta^{ij} - \frac{T^iT^j}{R^2} \tag{4.35}$$

and we get

$$\begin{aligned} \mathcal{L}_I = &-\frac{1}{2}\int d^3\theta\sqrt{g}\left(g^{\gamma\gamma'}\frac{T^kT^{k'}}{R^2} + g^{\gamma\beta'}g^{\gamma'\beta}\partial_\beta T^k\partial_{\beta'}T^{k'}\right)\partial_\gamma Y^k\partial_{\gamma'}Y^{k'} \\ &-\frac{1}{2}\int d^3\theta\sqrt{g}g^{\gamma\gamma'}\partial_\gamma Y^{\hat{k}}\partial_{\gamma'}Y^{\hat{k}}, \end{aligned}$$

[4] To see this we note that any vector in $\mathbb{R}^4$ can be written as

$$v^i = aT^i + b^\alpha\partial_\alpha T^i. \tag{4.34}$$

Then the identity can be proved by acting by both sides on this vector. We may also note the operator is a projector.

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$$\mathcal{L}_m = -\frac{m^2}{2}\int d^3\theta\sqrt{g}\left(Y^iY^i + Y^{\hat{i}}Y^{\hat{i}}\right) \tag{4.36}$$

We now proceed by inserting the expansions in terms of fluctuation fields defined as in eq. (4.3), which we repeat here,

$$\begin{aligned} Y^i &= \frac{T^i}{R}\phi^R + \phi^\alpha\partial_\alpha T^i, \\ Y^{\hat{i}} &= \phi^{\hat{i}} \end{aligned} \tag{4.37}$$

From this it follows that

$$\partial_\alpha Y^i = \frac{1}{R}(\partial_\alpha T^i)\phi^R + \frac{T^i}{R}\partial_\alpha\phi^R + (D^T_\alpha\phi^\beta)\partial_\beta T^i + \phi^\beta D^T_\alpha\partial_\beta T^i. \tag{4.38}$$

We have noted that $\partial_\alpha T^i$ transform as four vectors (one for each fixed value of i) on the three-sphere, or equivalently, that $\phi^\alpha\partial_\alpha T^i$ are four scalars on S^3. Consequently $\partial_\alpha(\phi^\beta\partial_\beta T^i) = (D^T_\alpha\phi^\beta)\partial_\beta T^i + \phi^\beta D^T_\alpha\partial_\beta T^i$. We then note that[5]

$$D^T_\alpha\partial_\beta T^i = -\frac{1}{R^2}g_{\alpha\beta}T^i \tag{4.39}$$

on S^3.

The main point in this paper is to express everything in terms of total derivatives. We motivate this by the fact that the supersymmetry parameter is constant only with respect to the total derivative. By noting that the only non-vanishing Christoffel symbols in our polar coordinate system are

$$\begin{aligned} &\Gamma^\gamma_{\alpha\beta}, \\ &\Gamma^R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{R} \\ &\Gamma^\alpha_{\beta R} = \frac{1}{R}\delta^\alpha_\beta \end{aligned} \tag{4.40}$$

We find that

$$\begin{aligned} D_\alpha\phi^\beta &= D^T_\alpha\phi^\beta + \frac{1}{R}\delta^\beta_\alpha\phi^R, \\ D_\alpha\phi^R &= \partial_\alpha\phi^R - \frac{1}{R}\phi_\alpha, \\ D_\alpha\phi^{\hat{i}} &= \partial_\alpha\phi^{\hat{i}}. \end{aligned} \tag{4.41}$$

Using all this, we find that

$$\partial_\alpha Y^i = (D_\alpha\phi^\beta)\partial_\beta T^i + \frac{T^i}{R}D_\alpha\phi^R. \tag{4.42}$$

Inserting this into $\mathcal{L}_{\text{pot}}$ we find the result

$$\begin{aligned} \mathcal{L}_{\text{pot}} = &-\frac{1}{2}D_\alpha\phi^\beta D_\beta\phi^\alpha - \frac{1}{2}g^{\alpha\beta}D_\alpha\phi^A D_\beta\phi^A \\ &-\frac{1}{2R^2}\phi^A\phi^A - \frac{1}{2R^2}g_{\alpha\beta}\phi^\alpha\phi^\beta. \end{aligned} \tag{4.43}$$

[5]The left-hand side is symmetric. Hence one can suspect the result be proportional to $g_{\alpha\beta}$ (or to the Ricci tensor, but these are proportional on S^3). The normalization is then fixed by computing $T^iD^T_\alpha\partial_\beta T^i = -\partial_\alpha T^i\partial_\beta T^i = -g_{\alpha\beta}$ where we used the three-sphere constraint $T^iT^i = R^2$ to move one derivative.

The placements of the derivatives in the first term looks funny and a naive guess could be that this is something ugly and unwanted. But in fact this precise juxtaposition of the two derivatives turns out to be crucial for getting a gauge invariant action. We can not make integration by parts using D_α since it does not lead to a total derivative. We rather have that a total derivative (which vanishes upon integration over closed three-sphere) is given by

$$\int d^3\theta\sqrt{g}D^T_\alpha V^\alpha = \int d^3\theta\partial_\alpha(\sqrt{g}V^\alpha). \tag{4.44}$$

So we must express everything in terms of the intrinsic covariant derivative D^T_α before we can make integrations by parts. We then find

$$\begin{aligned} D_\alpha\phi^\beta D_\beta\phi^\alpha &= D^T_\alpha\phi^\alpha D^T_\beta\phi^\beta + \phi^\beta[D^T_\alpha, D^T_\beta]\phi^\alpha \\ &\quad + \frac{2}{R}\phi^R D^T_\alpha\phi^\alpha + \frac{3}{R^2}(\phi^R)^2 \end{aligned} \tag{4.45}$$

On a three-sphere of radius R we have

$$[D^T_\alpha, D^T_\beta]\phi^\alpha = \frac{2}{R^2}\phi_\beta. \tag{4.46}$$

If then, we also expand out the other non-trivial term in $\mathcal{L}_{\text{pot}}$, which is

$$-\frac{1}{2}D_\alpha\phi^R D^\alpha\phi^R = -\frac{1}{2}D^T_\alpha\phi^R D^{T\alpha}\phi^R - \frac{1}{R}\phi^R D^T_\alpha\phi^\alpha - \frac{1}{2R^2}\phi^\alpha\phi_\alpha \tag{4.47}$$

then we find that the mass term $\phi^\alpha\phi_\alpha$ exactly cancels out in $\mathcal{L}_H$, and we end up with

$$\begin{aligned} \mathcal{L}_H &= -\frac{1}{2}D^T_\alpha\phi^\alpha D^T_\beta\phi^\beta - \frac{1}{2}D^T_\alpha\phi^A D^{T\alpha}\phi^A \\ &\quad -\frac{3}{2R}\phi^R D^T_\alpha\phi^\alpha - \frac{3}{2R^2}(\phi^R)^2 - \frac{1}{2R^2}\phi^A\phi^A \end{aligned} \tag{4.48}$$

Chern-Simons term. From the Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{1}{2}\epsilon^{\mu\nu\lambda}A_{\mu,ab}\partial_\nu A_{\lambda,cd}\left\langle T^c, \{T^b, T^c, T^d\}\right\rangle \tag{4.49}$$

we get

$$\mathcal{L}_{\text{CS}} = \frac{1}{2}\epsilon^{\mu\nu\lambda}g\epsilon^{\alpha\beta\gamma}\partial_\alpha B_{\mu\beta}\partial_\nu B_{\lambda\gamma} \tag{4.50}$$

Fermionic part. The fermionic part is

$$\frac{i}{2}\left\langle\bar\psi, \Gamma^\mu D_\mu\psi\right\rangle + \frac{i}{4}\left\langle\bar\psi, \Gamma_{ij}\{T^i, T^k, \psi\}\right\rangle - \frac{im}{2}\left\langle\bar\psi, \Sigma\Gamma_R\psi\right\rangle. \tag{4.51}$$

We expand the second term

$$\frac{i}{4}\left\langle\bar\psi, \Gamma_{ij}\{T^i, T^k, \psi\}\right\rangle = \frac{i}{2}\left\langle\bar\psi, \Sigma\Gamma^\alpha D_\alpha\psi\right\rangle. \tag{4.52}$$

We then make the field redefinition

$$\psi = U\chi,$$

$$\bar{\psi} = \bar{\chi} V \tag{4.53}$$

and we get

$$\frac{i}{2}\langle\bar{\chi},\Gamma^\mu\partial_\mu\chi\rangle + \frac{i}{2}\langle\bar{\chi},\Gamma^\alpha D_\alpha\chi\rangle - \frac{i}{2R}\langle\bar{\chi},\Sigma\Gamma_R\chi\rangle\,. \tag{4.54}$$

To get here we have used that

$$D_\alpha\Sigma = 0. \tag{4.55}$$

The induced Lagrangian

Summing up all the various contributions, the resulting induced six dimensional action that we obtain up to quadratic order, is given by

$$S = \int d^3x d^3\theta\sqrt{g}\left(\mathcal{L}_H + \mathcal{L}_\phi + \mathcal{L}_\psi\right) \tag{4.56}$$

where

$$\begin{aligned}\mathcal{L}_H =& -\frac{1}{12}g^{\alpha\beta\gamma,\alpha'\beta'\gamma'}H_{\alpha\beta\gamma}H_{\alpha'\beta'\gamma'} - \frac{1}{4}\eta^{\mu\mu'}g^{\alpha\beta,\alpha'\beta'}H_{\mu\alpha\beta}H_{\mu'\alpha'\beta'}\\ &-\frac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon^{\alpha\beta\gamma}\partial_\beta B_{\mu\alpha}\partial_\nu B_{\lambda\gamma} - \frac{1}{4R}\epsilon^{\alpha\beta\gamma}\phi^R H_{\alpha\beta\gamma},\end{aligned} \tag{4.57}$$

$$\begin{aligned}\mathcal{L}_\phi =& -\frac{1}{2}\left(\partial_\mu\phi^A\partial^\mu\phi^A + g^{\alpha\beta}D^T_\alpha\phi^A D^T_\beta\phi^A\right)\\ &-\frac{1}{2R^2}\phi^A\phi^A - \frac{3}{2R^2}\phi^R\phi^R\end{aligned} \tag{4.58}$$

$$\mathcal{L}_\psi = \frac{i}{2}\bar{\chi}\Gamma^\mu\partial_\mu\chi + \frac{i}{2}\bar{\chi}\Gamma^\alpha D^T_\alpha\chi + \frac{i}{4R}\bar{\chi}\Sigma\Gamma_R\chi. \tag{4.59}$$

For $\mathcal{L}_\psi$ we have used

$$D_\alpha\chi = D^T_\alpha\chi + \frac{1}{2R}\Gamma_R\Gamma_\alpha\Sigma\chi. \tag{4.60}$$

This follows from eq. (3.10) if one notes that $D_\alpha\Gamma_\beta = 0$ and $D^T_\alpha\Gamma_\beta = 0$. The first condition follows by requiring that $V_\beta = \bar{\psi}_1\Gamma_\beta\psi_2$ transforms like a vector for any two BLG spinors $\psi_{1,2}$. The second condition can be seen by requiring

$$D_\alpha V_\beta = D^T_\alpha V_\beta + \frac{1}{R}g_{\alpha\beta}V_R \tag{4.61}$$

If we assume that $D_\alpha\Gamma_\beta = 0$ then we get

$$D_\alpha V_\beta = (D_\alpha\bar{\psi}_1)\Gamma_\beta\psi_2 + \bar{\psi}_1\Gamma_\beta D_\alpha\psi_2 \tag{4.62}$$

We expand $D_\alpha\psi_{1,2} = D^T_{1,2}\psi_{1,2} - \frac{1}{2R}\Gamma_R\Gamma_\alpha\psi_{1,2}$, and we get

$$D_\alpha V_\beta = (D^T_\alpha\bar{\psi}_1)\Gamma_\beta\psi_2 + \bar{\psi}_1\Gamma_\beta D^T_\alpha\psi_2 + \frac{1}{R}g_{\alpha\beta}V_R \tag{4.63}$$

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and this equals $D^T_\alpha V_\beta + \frac{1}{R} g_{\alpha\beta} V_R$ only if

$$D^T_\alpha \Gamma_\beta = 0. \tag{4.64}$$

Using this, and also

$$\begin{aligned} D_\alpha g_{\beta\gamma} &= 0, \\ D^T_\alpha g_{\beta\gamma} &= 0 \end{aligned} \tag{4.65}$$

which can be seen as a consequence of $D_\alpha \Gamma_\beta = 0 = D^T_\alpha \Gamma_\beta$, or it can be derived direcctly as $D_\alpha g_{\beta\gamma} = D^T_\alpha g_{\beta\gamma} + \Gamma^R_{\alpha\beta} g_{R\gamma} + .. = D^T_\alpha g_{\beta\gamma}$ since $g_{R\alpha} = 0$ by our choice of coordinates. Of course $D^T_\alpha g_{\beta\gamma} = 0$ is the familiar metric compatibility condition. Taken this together we conclude that

$$\begin{aligned} D_\alpha U &= 0, \\ D^T_\alpha U &= 0 \end{aligned} \tag{4.66}$$

where U is defined as in eq. (4.23). we then get for any BLG spinor ψ related to χ as $\chi = U\psi$, $\psi = -U\chi$,

$$\begin{aligned} D_\alpha \chi &= U D_\alpha \psi \\ &= D^T_\alpha \chi + \frac{1}{2R} U \Gamma_R \Gamma_\alpha U \chi \\ &= D^T_\alpha \chi + \frac{1}{2R} \Gamma_R \Gamma_\alpha \Sigma \chi \end{aligned} \tag{4.67}$$

as asserted.

We note that the equation of motion for the two-form $B_{\mu\alpha}$ becomes a total derivative [7]. If then we vary $\partial_\beta B_{\mu\alpha}$, rather than $B_{\mu\alpha}$, then we find the equation of motion

$$\partial_\mu B_{\nu\alpha} - \partial_\nu B_{\mu\alpha} = -\frac{\sqrt{g}}{2} \epsilon_{\mu\nu\lambda} \epsilon_{\alpha\beta\gamma} H^{\lambda\beta\gamma} \tag{4.68}$$

This is the same equation of motion as we get directly from the BLG equation of motion eq. (2.20) by inserting our fluctuation expansion. To see this we first we contract eq. (2.20) by $T^a \partial_\alpha T^b$ and then insert the fluctuation field expansions into the resulting equation of motion. We also note the three-sphere constraint $T^i T^i = R^2$ which implies that $T^i D^\lambda \partial_\alpha T^i = -\partial_\alpha T^i D^\lambda T^i$.

The gauge field part of the lagrangian, $\mathcal{L}_H$, was also obtained in [6] and further studied in [7].

5 Induced supersymmetry

To get the supersymmetry variations we can expand the mass deformed BLG supersymmetry variations to linear order in the fluctuations. At zeroth order we have $\delta T^I = 0$. There are no 'higher order' contributions to the variation T^I since the higher order variations sit in the fluctuation fields $Y^I \equiv X^I - T^I$. At linear order we find the supersymmetry variations for the fluctuations as

$$\delta Y^I = i\bar{\epsilon}\Gamma^I \psi,$$

$$\begin{aligned}
\delta\psi &= \Gamma^\mu\Gamma_I\epsilon\partial_\mu T^I + m\Sigma\Gamma_R\Gamma_I\epsilon T^I - \Sigma\epsilon \\
&\quad -\Gamma^\mu\Gamma_I\epsilon A_{\mu,ab}\{T^a,T^b,T^I\} + \Gamma^\mu\Gamma_I\epsilon\partial_\mu Y^I \\
&\quad -\frac{1}{2}\Gamma_{IJK}\epsilon\{Y^I,T^J,T^K\} \\
&\quad +m\Sigma\Gamma_R\Gamma_I\epsilon Y^I, \\
\delta A_{\mu,ab} &= i\bar{\epsilon}\Gamma_\mu\Gamma_I T^I_{[a}\psi_{b]}
\end{aligned} \tag{5.1}$$

We can cancel the zeroth order contribution in $\delta\psi$ by taking T^I to lie on a three-sphere of constant radius $R = 1/m$. With this choice of radius we preserve maximal supersymmetry and the first line above in $\delta\psi$ vanishes.

Supersymmetry variations of the Bosons. From $\delta Y^I = \imath\bar{\epsilon}\Gamma^i\psi$ we get

$$\delta\phi^A = i\bar{\omega}\Gamma^A\chi \tag{5.2}$$

and

$$\delta B_{\alpha\beta} = i\bar{\omega}\Gamma_{\alpha\beta}\chi \tag{5.3}$$

and from $\delta A_{\mu,ab}$ we get

$$\delta B_{\mu\alpha} = i\bar{\omega}\Gamma_\mu\Gamma_\alpha\chi + \partial_\alpha\lambda_\mu \tag{5.4}$$

where

$$\lambda_\mu = \frac{i}{2}\bar{\epsilon}\Gamma_\mu\Gamma_R\psi \tag{5.5}$$

is a gauge parameter. We also note that

$$\lambda_\alpha = \frac{i}{2}\bar{\epsilon}\Gamma_\alpha\Gamma_R\psi \equiv 0 \tag{5.6}$$

so this is really a six-dimensional gauge parameter.

Supersymmetry variation of the Fermions. We insert the expansion eq. (4.3) and eq. (4.7) and get

$$\begin{aligned}
\delta\psi &= \Gamma^\mu\Sigma\Gamma^{\alpha\beta}\epsilon\partial_\alpha B_{\mu\beta} + \Gamma^\mu\Gamma_\alpha\epsilon\partial_\mu\phi^\alpha - \Sigma\epsilon D_\alpha\phi^\alpha \\
&\quad +\Gamma^\mu\Gamma_A\epsilon\partial_\mu\phi^A - \Gamma_A\Sigma\Gamma^\alpha\epsilon D_\alpha\phi^A \\
&\quad +\frac{1}{R}\Sigma\Gamma_R\Gamma_A\epsilon\phi^A - \frac{2}{R}\Sigma\Gamma_R\Gamma_\alpha\epsilon\phi^\alpha
\end{aligned} \tag{5.7}$$

We then dualize ϕ^α into $B_{\alpha\beta}$ and make a unitary rotation by means of the matrix U to gain six-dimensional covariance. We then get

$$\begin{aligned}
\delta\chi &= \frac{1}{2}\Gamma^\mu\Gamma^{\alpha\beta}H_{\mu\alpha\beta} + \frac{1}{6}\Gamma^{\alpha\beta\gamma}\omega H^D_{\alpha\beta\gamma} \\
&\quad -\Gamma^\mu\Gamma_A\omega\partial_\mu\phi^A - \Gamma^\alpha\Gamma_A\omega D_\alpha\phi^A \\
&\quad -\frac{1}{R}\Sigma\Gamma_R\Gamma_A\omega\phi^A + \frac{1}{2R}\Sigma\Gamma_R\Gamma^{\alpha\beta}\omega B_{\alpha\beta}.
\end{aligned} \tag{5.8}$$

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where we introduced

$$
\begin{aligned}
H^D_{\alpha\beta\gamma} &= D_\alpha B_{\beta\gamma} + D_\gamma B_{\alpha\beta} + D_\beta B_{\gamma\alpha} \\
&= H_{\alpha\beta\gamma} + \frac{1}{R}\sqrt{g}\epsilon_{\alpha\beta\gamma}\phi^R
\end{aligned} \tag{5.9}
$$

In terms of D^T_α derivatives we then find the result

$$
\begin{aligned}
\delta\chi = &\frac{1}{2}\Gamma^\mu\Gamma^{\alpha\beta}H_{\mu\alpha\beta} + \frac{1}{6}\Gamma^{\alpha\beta\gamma}\omega H_{\alpha\beta\gamma} \\
&-\Gamma^\mu\Gamma_A\omega\partial_\mu\phi^A - \Gamma^\alpha\Gamma_A\omega D^T_\alpha\phi^A \\
&-\frac{1}{R}\Sigma\Gamma_R\Gamma_A\omega\phi^A + \frac{1}{R}\Sigma\omega\phi^R.
\end{aligned} \tag{5.10}
$$

These supersymmetry variations must close on-shell on Lie derivatives on $\mathbb{R}^{1,2}\times S^3$, the $SO(4)\subset SO(5)$ R-symmetry that keeps ϕ^R fixed, and a gauge variation as these are the bosonic symmetries of the action.

6 Open problems

By taking k large we reduce S^3 to S^2 by shrinking the Hopf circle k times due to the $\mathbb{Z}_k$ orbifold identification, and the M5 brane wrapped on S^3 reduces to D4 wrapped on S^2. In would be interesting to demonstrate this explicitly in our abelian theory and make connection to [3]. Also since we know the nonabelian D4 brane theory this can give a hint of the nonabelian M5 brane theory.

One may consider more general mass deformations that still preserve maximal supersymmetry [26]. It would be interesting to see what M5 brane theories these correspond to. We may also get less supersymmetric six-dimensional theories by expanding BLG theory about less supersymmetric backgrounds, such as has been classified in [25]. In particular one can consider the half BPS funnel solution [24] of M2's ending on M5 and find a six dimensional theory with eight supercharges on curved manifold of the geometry of a funnel.

The right way to discretize the BLG theory with a Nambu bracket should be to consider ABJM theory. Needless to say it will be very interesting to derive BLG theory on $S^3/\mathbb{Z}_k$ by taking the large N limit of mass deformed ABJM theory at level k. For $k=1,2$ we can indeed see the fuzzy three-sphere (mod $\mathbb{Z}_2$) in ABJM theory. Only for levels $k=1,2$ do we have enhanced SO(8) R symmetry in ABJM theory for generic gauge groups. In this case we can find a fuzzy funnel solution that is locally a fuzzy three-sphere. We have not yet verified that a similar type of enhancement works also for the mass deformed ABJM theory eventhough this seems very plausible, so let us demonstrate how the fuzzy funnel solution arises. For levels $k=1,2$ we have showed in [17] that the supersymmetry variation of the fermion in ABJM theory can be written as (using the same notations as in that paper)

$$
\delta\psi = \Gamma^\mu\Gamma_I\epsilon D_\mu X^I - \frac{1}{6}\Gamma_I\Gamma_J\Gamma_K\epsilon[X^I,X^J;X^K]. \tag{6.1}
$$

Moreover we can antisymmetrize IJK despite the three-bracket is only manifestly antisymmetric in its first two entries. This follows from the identity

$$
X^I_b X^J_c X^{Kd} f^{bc}{}_{da} = X^K_b X^{[I}_c X^{J]d} f^{bc}{}_{da} \tag{6.2}
$$

We then find that

$$\begin{aligned}\Gamma_{IJ}[X^I, X^J; X^K]_a &\equiv \Gamma_{IJ} X_b^I X_c^J X^{Kd} f^{bc}{}_{da} \\ &= \Gamma_{IJ} X_b^K X_c^I X^{Jd} f^{bc}{}_{da} \\ &= -\Gamma_{IJ} X_b^I X_c^K X^{Jd} f^{bc}{}_{da}\end{aligned} \tag{6.3}$$

Hence the bracket can be antisymmetrized in I, J, K when contracted by Γ_{IJ}. From here we can then derive the Basu-Harvey fuzzy three-sphere funnel solution [24] by requiring $\delta\psi = 0$. For level $k = 2$ the $\mathbb{Z}_2$ orbifolding is just $X^I \sim -X^I$ that comes from $Z_a^A \sim -Z_a^A = e^{i\pi} Z_a^A$. The relation between X_a^I and Z_a^A involves a Wilson line W_{ab} and the higher k orbifolding $Z_a^A \sim e^{2\pi i/k} Z_a^A$ has no such simple counterpart for the X_a^I. Also the Wilson line becomes non-local and it is unclear to us whether one can find a fuzzy three-sphere mod $\mathbb{Z}_k$ also for higher levels k.

In [9] it was demonstrated how the Nambu-Goto action for a five-brane can be reformulated as a BLG type of theory with a Nambu three-bracket. It will be interesting to generalize this approach to the full-fledged kappa symmetric M5 brane action [23] and derive (mass deformed) BLG theory from this action.

In principle the theory of multiple M5 branes should also be encoded in some ABJM theory. It would be very interesting to see if one can compute any quantity in the multiple M5 brane theory from ABJM theory. For finite rank gauge groups we would expect to find a non-commutative, and perhaps also non-abelian, M5 brane.

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A Gamma matrix relations

For the matrices

$$\begin{aligned}U &= \frac{i}{\sqrt{2}}\tilde{\Gamma}(1-\Sigma), \\ V &= -\frac{i}{\sqrt{2}}(1-\Sigma)\tilde{\Gamma}.\end{aligned} \tag{A.1}$$

we have used the following identities,

$$\begin{aligned}U\Sigma\Gamma_R\Gamma_\alpha U &= \Gamma_R\Gamma_\alpha \\ U\Gamma^\mu\Sigma\Gamma^{\alpha\beta}U &= \Gamma^\mu\Gamma^{\alpha\beta} \\ U\Gamma^{\alpha\beta\gamma}U &= \Gamma^{\alpha\beta\gamma} \\ U\Gamma^\mu\Gamma_A U &= \Gamma^\mu\Gamma_A \\ U\Gamma^\alpha\Gamma_A\Sigma U &= \Gamma^\alpha\Gamma_A \\ U\Sigma U &= \Sigma.\end{aligned} \tag{A.2}$$

and

$$
\begin{aligned}
V\Gamma^{\mu}U &= \Gamma^{\mu}, \\
V\Sigma\Gamma^{\alpha}U &= \Gamma^{\alpha}, \\
V\Gamma_{\mu}\Gamma_{\alpha}U &= -\Gamma_{\mu}\Gamma_{\alpha}, \\
V\Gamma_{A}U &= -\Gamma_{A}, \\
V\Gamma_{\alpha}U &= \Gamma_{\alpha}\Sigma \\
V\Sigma\Gamma_{\alpha\beta}U &= -\Gamma_{\alpha\beta}.
\end{aligned} \tag{A.3}
$$

Our gamma matrices are subject to the algebra

$$
\begin{aligned}
\{\Sigma, \tilde{\Gamma}\} &= 0, \\
\{\Gamma_{\mu}, \Sigma\} &= 0, \\
[\Gamma_{\alpha}, \Sigma] &= 0, \\
\{\Gamma_{A}, \Sigma\} &= 0, \\
[\Gamma_{\mu}, \tilde{\Gamma}] &= 0, \\
\{\Gamma_{\alpha}, \tilde{\Gamma}\} &= 0, \\
\{\Gamma_{A}, \tilde{\Gamma}\} &= 0, \\
\{\Gamma_{\mu}, \Gamma_{\alpha}\} &= 0, \\
\{\Gamma_{\mu}, \Gamma_{A}\} &= 0, \\
\{\Gamma_{\alpha}, \Gamma_{A}\} &= 0.
\end{aligned} \tag{A.4}
$$

and duality relations

$$
\begin{aligned}
\Sigma\Gamma^{\gamma} &= \frac{1}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma}\Gamma_{\alpha\beta}, \\
\Gamma^{\gamma} &= -\frac{1}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma}\Sigma\Gamma_{\alpha\beta}, \\
\Gamma_{\gamma} &= -\frac{1}{2}\sqrt{g}\epsilon_{\alpha\beta\gamma}\Sigma\Gamma^{\alpha\beta}, \\
\Gamma_{\gamma}\epsilon^{\alpha\beta\gamma} &= -\frac{1}{\sqrt{g}}\Sigma\Gamma^{\alpha\beta}.
\end{aligned} \tag{A.5}
$$

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